which can be obtained by integration of the well-known equation

$$\frac{\partial c_+}{\partial y_+} = \frac{1}{Pr^{-1} + b y_+^n}, \quad c_+ = \frac{(c - c_w)u_*}{j_w}, \tag{3}$$

where in accordance with the experimental data from ref. [2] it is assumed that $n = 3$. The coefficient b in equation (3) is assumed to be equal to 1/1320; this value of b lies within the limits for b-values determined in ref. [2] for $Re \gtrsim 16 \times 10^3$ and it leads to $A = 1$, which simplifies the calculations by formula (2).

A large spread of the experimental points in Fig. 1 apparently indicates that the turbulent structure in the viscous sublayer depends on Re in the studied range of small Reynolds numbers. This dependence is especially noticeable at $Re \lesssim 10^4$, but it is not negligible up to the maximum values of Re reached in the experiments. These conclusions agree with the results of the temperature profile measurements presented in ref. [3], where it was found that for $Re \lesssim 25 \times 10^3$ the additive term in the logarithmic temperature distribution law, which represents the temperature drop in a viscous sublayer, depends not only on Pr, but also on Re. The presented data also show that the values of Re, at which the concentration profile can be considered as stabilized, were not attained in the considered experiments. This remark is even more valid for the $c(y)$ profiles measured in ref. [2] at $Pr = 2140$ and $Re \leqslant 10^4$, which for this very reason, are not considered here.

Figure 1 shows also that at $Re \gtrsim 20 \times 10^3$ the measured concentration profile is satisfactorily described both by the limiting relation (1) and equation (2). It should be noted, however, that best agreement is obtained at $Re = 16 \times 10^3$, while at $Re = 25 \times 10^3$ the agreement is somewhat worse.

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ON A SLOWLY EVAPORATING DROPLET NEAR A HOT PLATE

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NOMENCLATURE

c	specific heat of vapor
h	height of droplet above the plate
L	latent heat of vaporization
p	pressure
R	radius of droplet
T	temperature
T_s	temperature of droplet
T_w	plate temperature
T_∞	ambient temperature
$\mathbf{u}$	velocity vector.

Greek symbols

α	thermal diffusivity
λ	thermal conductivity
ν	kinematic viscosity
ρ	density
ψ	streamfunction.

Subscripts

n	normal component
t	tangential component.

1. INTRODUCTION

THE PHENOMENON of a liquid droplet evaporating near a hot surface arises in many practical applications. For instance, in compression–ignition engines, gas turbines and other oil-fired appliances, fuel droplets are injected onto the hot walls of the combustion chamber where they must vaporize, prior to ignition. In the so-called open-chamber, stratified charge engines, the liquid fuel is directly injected into a hollow space in the piston, in the form of droplets or sprays. Evaporation of the droplets within this hollow space and subsequent combustion is known to improve engine performance.

Another situation involving vaporization of fuel droplets over a hot surface is that of fumigation. Here a small fraction of the fuel is deliberately injected onto the walls of the intake manifold. This fuel fraction evaporates and thereby reduces the delay in the formation of a combustible mixture and minimizes the rate of the pressure rise in the cylinder. Similar configurations may occur under fire fighting conditions where water droplets are injected to impinge on a hot ablating surface.

In view of the many applications, an understanding of the behavior of a liquid droplet evaporating near a hot surface is of fundamental importance. A systematic study of these phenomena began with the experimental observations of Tamura and Tanasawa [1] who presented a detailed description of the different modes of evaporation including the levitated spheroidal state. To be sure, similar observations were made much earlier by Johann Leidenfrost (1756) who reported dancing of water droplets over a red-hot iron spoon. Here a continuous vapor film separates the vaporizing liquid from the hot surface and the liquid droplet is suspended above the hot surface by its own vapor pressure. This phenomenon, known subsequently as the Leidenfrost phenomenon, is considered a special case of film boiling and is sometimes referred to as a small-scale vapor explosion [2].

There have been numerous attempts to give an analytical description of the Leidenfrost phenomenon. The first such attempt was due to Pleteneva and Rebinder [3] who presented a semianalytical solution to the problem. Among others, Gorton [4] postulated a model in which the vapor flow was assumed to be potential. Kistemaker [5] tried to improve on Gorton's work by dealing with the Navier–Stokes equations to describe the vapor flow below the droplet. Gottfried *et al.* [6] criticizes this solution (which is erroneous) and gives an overview of the previous attempts. Later, Satcunanathan [7] formulated a simple model to calculate the evaporation rate of a droplet next to a hot plate. Recently, Yang [8] has analyzed the evaporation process for the contact evaporation regime.

In their pioneering work, Gottfried *et al.* [6] presented a mechanistic analysis of the problem. They studied the behavior of a small droplet with no bouncing effects and developed an analytical model to predict the rate of evaporation, among other things. They presumed, for the first time, that the essential mechanism of the Leidenfrost phenomenon was film boiling of a liquid phase whose quantity was insufficient to completely cover the heating surface. The liquid existed in small masses separated from the hot surface by a film of vapor generated by evaporation of the liquid. The masses were supported by the pressure gradient in the vapor flowing underneath the mass.

Although mechanistically correct, the work of Gottfried *et al.* [6] lacks the rigor of an analytical treatment. The motivation of the present work comes from providing a rational analytical approach to the droplet evaporation problem. This paper is the first of a series of investigations on the subject. Here we examine the droplet vaporization process next to a hot plate in the absence of convective and radiative transfer. We assume the droplet surface temperature to be prescribed *a priori*. Neglecting gravity, we will calculate the force on the droplet due to vapor pressure and study its variation with a change in the plate temperature and the distance of the droplet from the plate. Further assumptions pertinent to the analysis are given in Section 2.

2. MATHEMATICAL FORMULATION

Consider a spherical liquid droplet near a hot plate of infinite extent as shown in Fig. 1. The plate is maintained at a temperature T_w which is significantly higher than the temperature T_s of the droplet. Due to heat transfer from the plate, the droplet vaporizes. As a result of the evaporation process, the droplet experiences a force due to vapor pressure. As mentioned in the introduction, the droplet is assumed to be at a uniform surface temperature T_s which is known, *a priori*. To simplify the analysis further, we make the following assumptions:

(1) The shear induced circulation within the droplet may be neglected and the controlling resistance to heat and mass transfer is in the vapor phase.

(2) The temporal variation of velocity and temperature in the vapor is much smaller compared with their spatial variations.

(3) The vapor is a viscous Newtonian fluid whose motion is governed by the Navier–Stokes equations, in the creeping flow limit.

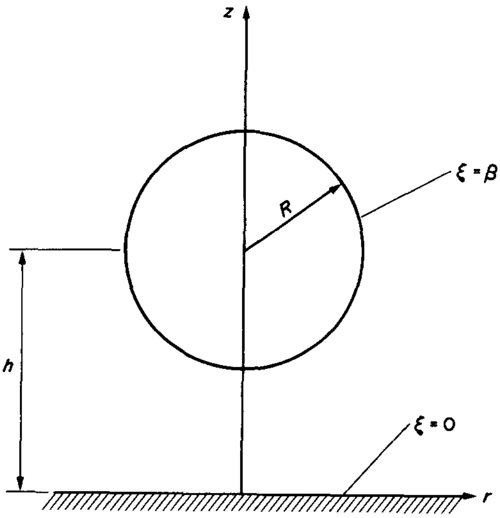

FIG. 1. Schematic diagram showing a droplet near a hot plate.

(4) Droplet evaporation may be considered a quasi steady state process, i.e. the evaporation process is slow enough that the velocity and temperature fields would be insignificantly different from those which would exist if the droplet were not changing its size. The validity of this assumption is discussed in detail by Ranz [9].

(5) The droplet retains its spherical shape throughout the evaporation process.

(6) Heat transfer due to radiation is ignored.

(7) Effect of gravity is neglected. (This effect will be included in a later paper to show that the droplet can be suspended near the plate due to a balance of buoyancy and vapor pressure forces.)

Under these assumptions, the equations of momentum and energy conservation in the vapor phase are given by

$$\mathbf{u}\cdot\nabla\mathbf{u} = -\rho^{-1}\nabla p + \nu\nabla^2\mathbf{u}, \tag{1a}$$

$$\mathbf{u}\cdot\nabla T = \alpha\nabla^2 T. \tag{1b}$$

These equations are subject to the following boundary conditions. On the plate $z = 0$, we have

$$\mathbf{u} = 0, \quad T = T_w. \tag{2}$$

On the surface of the droplet $r = R$, the conditions are

$$u_t = 0, \quad T = T_s, \quad \lambda\, \partial T/\partial n = \rho u_n L. \tag{3}$$

This last condition relates the heat and mass fluxes at the droplet surface through the latent heat of vaporization. Far away from the droplet, as $r \to \infty$, the velocity and temperature fields must satisfy

$$\mathbf{u} \to 0, \quad T \to T_\infty. \tag{4}$$

In two component systems, as the present one, equations (1a) and (1b) should be augmented by a conservation equation for the diffusing component. Furthermore, the surface temperature of the droplet cannot be prescribed *a priori* and must be determined as part of the solution. In this case, the boundary condition, equation (3b), must be replaced by an appropriate form of the Clausius–Clapeyron equation, relating the unknown surface temperature of the droplet to the concentration of the diffusing species at the droplet surface. This problem will be treated in another paper.

For the present purpose, it is convenient to use bipolar coordinates (ξ, η, ϕ) which are related to cylindrical coordinates (r, z, ϕ) through the transformations

$$z = \frac{a \sinh \xi}{\cosh \xi - \cos \eta}, \quad r = \frac{a \sin \eta}{\cosh \xi - \cos \eta}, \tag{5}$$

with $a > 0$. In this bipolar system, the surfaces generated by rotating the curves ξ = const. about the z-axis are a family of spheres of radii a cosech ξ, having the plate $z = 0$ (or $\xi = 0$) as a common radical plane. The spherical droplet can be located by the level surface $\xi = \beta$ (β being a positive constant). For the present configuration, the radius of the droplet and the distance of its center from the plate are given by [10]

$$R = a \operatorname{cosech} \beta, \quad h = a \coth \beta. \tag{6}$$

We now nondimensionalize all variables using the following scales:

$u_* = \lambda\rho c a$ for velocity,

$p_* = \rho\nu u_*/a$ for pressure,

$t_* = L/c$ for temperature.

The characteristic length scale 'a' can be easily derived from equation (6). With these scales, the dimensionless equations become

$$\bar{R}\mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \nabla^2\mathbf{u}, \tag{7a}$$

$$\bar{P}\mathbf{u}\cdot\nabla T = \nabla^2 T, \tag{7b}$$

where

$$\bar{R} = u_* a/\nu, \quad \bar{P} = u_* a/\alpha. \tag{8}$$

For small liquid droplets say, of water, we can assume that both the Reynolds number and the Peclet number are sufficiently small compared with unity. As a result, the inertial terms in the momentum equations and the convective terms in the energy equation can be ignored to a good approximation. In view of the axial symmetry of the resulting problem, it is appropriate to introduce a Stokes streamfunction ψ defined by

$$u_\xi = \frac{(\cosh\,\xi - \cos\,\eta)^2}{\sin\,\eta}\frac{\partial\psi}{\partial\eta}, \quad u_\eta = \frac{(\cosh\,\xi - \cos\,\eta)^2}{\sin\,\eta}\frac{\partial\psi}{\partial\xi}. \tag{9}$$

The momentum equations now reduce to

$$E^4\psi = 0, \tag{10}$$

where

$$E^2 = (\cosh\,\xi - \mu)\left[\frac{\partial}{\partial\xi}\left\{(\cosh\,\xi - \mu)\frac{\partial}{\partial\xi}\right\} + (1-\mu^2)\frac{\partial}{\partial\mu}\left\{(\cosh\,\xi - \mu)\frac{\partial}{\partial\mu}\right\}\right], \tag{11}$$

with

$$\mu = \cos\,\eta. \tag{12}$$

The energy equation is simply given by

$$\nabla^2 T = 0. \tag{13}$$

Note that T now measures temperature in excess of T_w. The dimensionless boundary conditions, equations (2)–(4) become

$$\xi = 0: \quad \psi = \partial\psi/\partial\xi = 0, \quad T = 0, \tag{14}$$

together with

$$\xi = \beta: \quad \partial\psi/\partial\xi = 0, \quad T = T_s - T_w, \quad \partial T/\partial\xi = -(\cosh\,\beta - \mu)\partial\psi/\partial\mu, \tag{15}$$

and

$$r \to \infty: \quad \mathbf{u} \to 0, \quad T \to T_\infty - T_w. \tag{16}$$

It is well known [11] that the solution of equation (13) satisfying condition (16) can be written in the form

$$T = (T_\infty - T_w) + (\cosh\,\xi - \mu)^{1/2}\sum_{n=0}^{\infty} G_n(\xi)P_n(\mu), \tag{17}$$

where

$$G_n(\xi) = A_n \cosh\,(n+1/2)\xi + B_n \sinh\,(n+1/2)\xi. \tag{18}$$

The constants A_n and B_n are determined by use of the boundary conditions, equations (14c) and (15b). We find

$$A_n = \sqrt{2}(T_w - T_\infty), \tag{19}$$

and

$$B_n = \sqrt{2}[(T_s - T_\infty)\,\mathrm{e}^{-(n+1/2)\beta} - (T_w - T_\infty)\cosh\,(n+1/2)\beta]/\sinh\,(n+1/2)\beta. \tag{20}$$

The general solution of equation (10) which vanishes at infinity is given by [12]

$$\psi = (\cosh\,\xi - \mu)^{-3/2}\sum_{n=0}^{\infty} C_{n+1}^{-1/2}(\mu) \times [a_n \cosh\,(n-1/2)\xi + b_n \sinh\,(n-1/2)\xi + c_n \cosh\,(n+3/2)\xi + d_n \sinh\,(n+3/2)\xi]. \tag{21}$$

Here $C_n(\mu)$ represents a Gegenbauer polynomial The constants a_n, b_n, c_n and d_n are determined as follows. Equations (14a) and (14b) give, respectively

$$a_n + c_n = 0, \quad (2n-1)b_n + (2n+3)d_n = 0. \tag{22}$$

Finally, the boundary conditions, equations (15a) and (15c), yield, after some manipulation, the following recursive system for the unknown constants a_n and b_n

$$\begin{bmatrix} a_{n-1} \\ b_{n-1} \end{bmatrix} = [C]^{-1}\left\{\begin{bmatrix} 0 \\ R_n + S_n \end{bmatrix} - [A]\begin{bmatrix} a_{n+1} \\ b_{n+1} \end{bmatrix} - [B]\begin{bmatrix} a_n \\ b_n \end{bmatrix}\right\}, \tag{23}$$

where

$$R_n = \frac{1}{3}\sinh\,\beta\left[\frac{n+1}{2n+3}G_{n+1}(\beta) + \frac{n}{2n-1}(\beta) - \gamma G_n(\beta)\right],$$

$$S_n = -\frac{2}{3}\gamma^2 G_n'(\beta) + \frac{4}{3}\gamma\left[\frac{n+1}{2n+3}G_{n+1}'(\beta) + \frac{n}{2n-1}G_{n-1}'(\beta)\right] - \frac{1}{3}\left[\frac{2(n+1)(n+2)}{(2n+3)(2n+5)}G_{n+2}'(\beta) + \frac{2(2n^2+2n-1)}{(2n-1)(2n+3)}G_n'(\beta)\right],$$

$$[A] = \begin{bmatrix} \dfrac{2n}{3(2n+3)}\dfrac{r_{n+1}}{\sinh\,\beta} & \dfrac{2n}{3(2n+3)}\dfrac{S_{n+1}}{\sinh\,\beta} \\ \dfrac{2n+5}{3(2n+3)}p_{n+1} & \dfrac{2n+5}{3(2n+3)}q_{n+1} \end{bmatrix}, \tag{24}$$

$$[B] = \begin{bmatrix} p_n - \dfrac{2}{3}r_n \coth\,\beta & q_n - \dfrac{2}{3}s_n \coth\,\beta \\ -\dfrac{2}{3}\gamma p_n & -\dfrac{2}{3}\gamma q_n \end{bmatrix}.$$

and

$$[C] = \begin{bmatrix} \dfrac{2(n+1)}{3(2n-1)}\dfrac{r_{n-1}}{\sinh\,\beta} & \dfrac{2(n+1)}{3(2n-1)}\dfrac{S_{n-1}}{\sinh\,\beta} \\ \dfrac{2n-3}{3(2n-1)}p_{n-1} & \dfrac{2n-3}{3(2n-1)}q_{n-1} \end{bmatrix}. \tag{25}$$

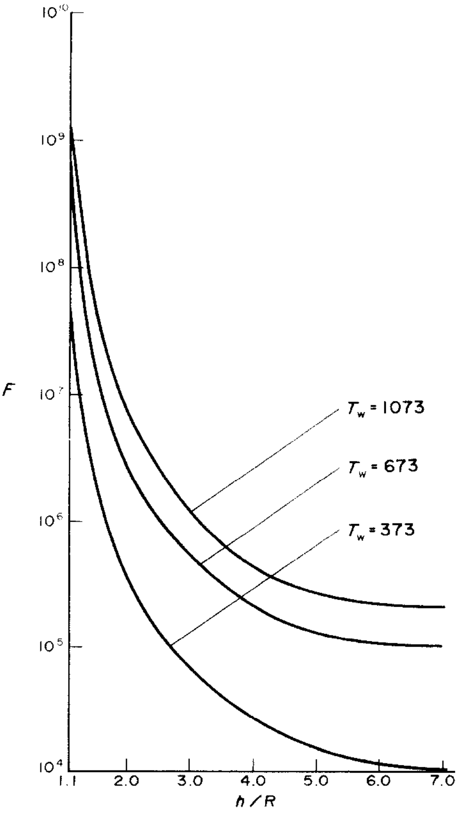

FIG. 2. Variation of the force on the droplet with h/R ratio for various plate temperatures.

with

$$\gamma = \cosh \beta.$$

Here

$$p_n = \cosh\,(n-1/2)\beta - \cosh\,(n+3/2)\beta,$$

$$q_n = \sinh\,(n-1/2)\beta - \left(\frac{2n-1}{2n+3}\right) \sinh\,(n+3/2)\beta,$$

$$r_n = (n-1/2)\left[\sinh\,(n-1/2)\beta - \left(\frac{2n+3}{2n-1}\right) \sinh\,(n+3/2)\beta\right],$$

$$s_n = (n-1/2)[\cosh\,(n-1/2)\beta - \cosh\,(n+3/2)\beta]. \qquad (26)$$

Clearly for a numerical evaluation of the constants a_{n-1} and b_{n-1}, the values of a_n, b_n, a_{n+1} and b_{n+1} must be known. We note from equation (21) that $a_n \to 0$ and $b_n \to 0$ exponentially with $n \to \infty$, in order that the velocity field remains bounded. Thus, by setting $a_N = b_N = a_{N+1} = b_{N+1} = 0$ for a sufficiently large value of $n = N$, the coefficients a_0 to a_{N-1} and b_0 to b_{N-1} can be computed using equation (23). For practical computations, different values of N are chosen until the lower order constants agree to within specified limits of accuracy for two choices of N.

The dimensionless force on the droplet in the positive z-direction can be given by [11,12]

$$F = -2\sqrt{2}\pi \sum_{n=0}^{\infty} (a_n + b_n + c_n + d_n). \qquad (27)$$

In view of the relations (22), this force has the value

$$F = -8\sqrt{2}\pi \sum_{n=0}^{\infty} \frac{1}{2n+3} b_n. \qquad (28)$$

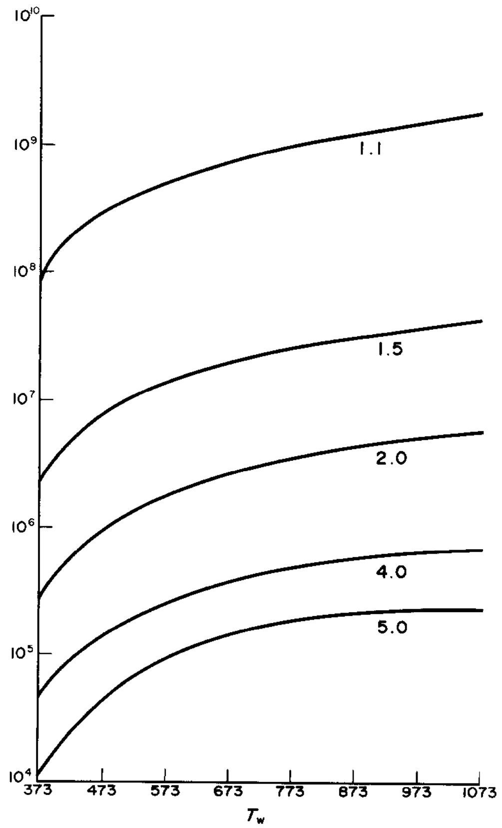

FIG. 3. Variation of the force on the droplet with the plate temperature for various h/R ratios.

3. RESULTS AND DISCUSSION

For a fixed value of β (i.e. for a fixed h/R ratio) and for given values of T_s, T_w and T_∞, the recursive system of equations (23) is solved for the constants a_n and b_n. Using the values of b_n thus obtained in equation (28), the force F on droplet is found. Next the value of β is changed (keeping T_s, T_w and T_∞ fixed) and the force is computed. This process is repeated for several values of β, to study the variation of F with the h/R ratio. The results are shown in Fig. 2, for a water droplet. Clearly the force increases with a decrease in the h/R ratio, this increase being significantly more pronounced at small gaps. Thus for a droplet of a given size (radius R) the force on the droplet increases as it gets closer to the plate. When the droplet is sufficiently close to the plate, the force on the droplet becomes exceedingly large. This is because at small gaps, large pressures must develop in the vapor phase in order to cause the vapor film to flow radially outwards. In fact, it is known that the droplet would experience an infinitely large force when it is in contact with the plate which is impermeable [11].

The effect of the variation in plate temperature on the force on the droplet for a fixed h/R ratio is shown in Fig. 3. The figure shows that for a given h/R ratio, the force on the droplet increases as the plate temperature is increased.

4. CONCLUDING REMARKS

The problem of a slowly evaporating droplet near a hot plate has been analyzed. The droplet is assumed to be at a uniform surface temperature which is known *a priori*. The force on the droplet is determined by examining the motion in the vapor phase, in the absence of convective and radiative effects. It is found that for a fixed plate temperature, the force on a droplet of a given radius increases as the droplet gets closer to the plate. This increase becomes much more pronounced at sufficiently small gaps. We also found that for a fixed location of the droplet of any given size, the force on the droplet increases as the temperature of the plate is raised.

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